

Random process. Consider a random experiment specified by the outcomes ζ from some sample space S , the events defined on S , and the probabilities on S . Suppose that to every outcome $\zeta \in S$, we assign a function of time according to some rule $X(t, \zeta), t \in I$. The graph of $X(t, \zeta)$ vs. t , for t fixed, is a **realization** of the RP. For each fixed t_k from the index set I , $X(t_k, \zeta)$ is a RV. RP is an indexed family of RVs.

Mean. $m_X(t) = E[X(t)] = \int_{\mathbb{R}} xf_{X(t)}(x)dx$, for $f_{X(t)}$ the pdf of $X(t)$.

Autocorrelation. $R_X(t_1, t_2) = E[X(t_1)X(t_2)] = \int_{\mathbb{R}} \int_{\mathbb{R}} xyf_{X(t_1)}f_{X(t_2)}(x, y)dxdy$.

Cross-correlation: $R_{X,Y}(t_1, t_2) = E[X(t_1)Y(t_2)]$; X, Y **orthogonal** if $R_{X,Y}(t_1, t_2) = 0 \forall t_1, t_2$.

Autocovariance. $C_X(t_1, t_2) = E[\{X(t_1) - m_X(t_1)\}\{X(t_2) - m_X(t_2)\}] = R_X(t_1, t_2) - m_X(t_1)m_X(t_2)$; $\text{var}[X(t)] = C_X(t, t)$. **Cross-covariance:** $C_{X,Y}(t_1, t_2) = E[\{X(t_1) - m_X(t_1)\}\{Y(t_2) - m_Y(t_2)\}] = R_{X,Y}(t_1, t_2) - m_X(t_1)m_Y(t_2)$; X, Y **uncorrelated** if $C_{X,Y}(t_1, t_2) = 0 \forall t_1, t_2$.

X, Y **independent** if vector RVs $(X(t_1), \dots, X(t_k))$ and $(Y(t'_1), \dots, Y(t'_j))$ independent $\forall k, j$, choices of t_1, \dots, t_k and t'_1, \dots, t'_j . If X, Y Gaussian, X, Y independent $\Leftrightarrow X, Y$ uncorrelated.

$X(t)$ **stationary** if $F_{X(t_1), \dots, X(t_k)}(x_1, \dots, x_k) = F_{X(t_1+\tau), \dots, X(t_k+\tau)}(x_1, \dots, x_k) \forall \tau, k, \{t_1, \dots, t_k\}$. Also, $m_X(t) = E[X(t)] = m \forall t$; $\text{var}[X(t)] = E[(X(t) - m)^2] = \sigma^2 \forall t$; $R_X(t_1, t_2) = R_X(t_2 - t_1) \forall t_1, t_2$; $C_X(t_1, t_2) = C_X(t_2 - t_1) \forall t_1, t_2$.

$X(t)$ **wide-sense stationary** (WSS) if $m_X(t) = m \forall t$ and $R_X(t_1, t_2) = R_X(t_2 - t_1) \forall t_1, t_2$. **Average power** (second moment): $E[X^2(t)] = R_X(0) \forall t$. Also $R_X(\tau) = R_X(-\tau)$; $P[|X(t + \tau) - X(t)| > \epsilon] = \frac{2}{\epsilon^2}[R_X(0) - R_X(\tau)]$; $\lim_{\tau \rightarrow \infty} R_X(\tau) = E[X(t)]^2$; and $|R_X(\tau)| \leq R_X(0)$; $R_X(0) = R_X(d) \Rightarrow R_X(\tau)$ is periodic with period d and $X(t)$ is **mean square periodic**: $E[X(t + d) - X(t)]^2 = 0$. $X(t)$ Gaussian, WSS $\Rightarrow X(t)$ stationary. $X(t)$ stationary $\Rightarrow X(t)$ WSS.

$X(t)$ **mean-square continuous** at t_0 if $E[(X(t) - X(t_0))^2] \rightarrow 0$ as $t \rightarrow t_0$; i.e., l.i.m. $_{t \rightarrow t_0} X(t) = X(t_0)$; if mean-square limit exists, **mean-square derivative** is $X'(t) = \frac{dX(t)}{dt} = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon}[X(t + \epsilon, \zeta) - X(t, \zeta)]$. $X'(t)$ exists at t if $\frac{\partial^2}{\partial t_1 \partial t_2} R_X(t_1, t_2)$ exists at $(t_1, t_2) = (t, t)$; for $X(t)$ WSS, $X'(t)$ exists if $\frac{d^2}{d\tau^2} R_X(\tau)$ exists at $\tau = 0$. For $X(t)$ Gaussian, $X'(t)$ also Gaussian, if it exists. $E[X'(t)] = \frac{d}{dt} m_X(t)$ ($= 0$ if $X(t)$ WSS). $R_{X'}(t_1, t_2) = \frac{\partial^2}{\partial t_1 \partial t_2} R_X(t_1, t_2)$ ($= -\frac{d^2}{d\tau^2} R_X(\tau)$ if $X(t)$ WSS). **Mean-square integral** is $Y(t) = \int_{t_0}^t X(t')dt' = \text{l.i.m.}_{\Delta_k \rightarrow 0} \sum_k X(t_k)\Delta_k$. $m_Y(t) = \int_{t_0}^t m_X(t')dt'$ and $R_Y(t_1, t_2) = \int_{t_0}^{t_1} \int_{t_0}^{t_2} R_X(u, v)dudv$, and $Y(t)$ exists if $R_Y(t_1, t_2)$ exists.

Power spectral density. For $X(t)$ WSS, $S_X(f) = \mathcal{F}\{R_X(\tau)\} = \int_{\mathbb{R}} R_X(\tau)e^{-j2\pi f\tau}d\tau$. For $X(t) \in \mathbb{R}$, $R_X(\tau) = R_X(-\tau)$, so $S_X(f) =$

$\int_{\mathbb{R}} R_X(\tau) \cos 2\pi f\tau d\tau = S_X(-f) \geq 0$. $R_X(\tau) = \mathcal{F}^{-1}\{S_X(f)\} = \int_{\mathbb{R}} S_X(f)e^{j2\pi f\tau}df$. **Average power** of $X(t)$: $E[X^2(t)] = R_X(0) = \int_{\mathbb{R}} S_X(f)df$. **Cross-power spectral density:** $S_{X,Y}(f) = \mathcal{F}\{R_{X,Y}(\tau)\}$, where $R_{X,Y}(\tau) = E[X(t + \tau)Y(t)]$. For $X(t)$ input into a linear system with impulse response $h(t)$ and output $Y(t)$, $S_Y(f) = |H(f)|^2 S_X(f)$.

$X(t)$ **Markov** if for arbitrary times $t_1 < t_2 < \dots < t_k < t_{k+1}$, $P[X(t_{k+1}) = x_{k+1} | X(t_k) = x_k, \dots, X(t_1) = x_1] = P[X(t_{k+1}) = x_{k+1} | X(t_k) = x_k]$. $X(t)$ is the **state** of the process at time t . **Markov chain** is an integer-valued Markov process. If $P[X_{n+1} = j | X_n = i] = p_{ij} \forall n$, X_n has **homogeneous transition probabilities** and $P[X_n = i_n, \dots, X_0 = i_0] = p_{i_{n-1}, i_n} \dots p_{i_0, i_1} p_{i_0}(0)$. X_n is completely specified by the initial pmf $p_i(0)$ and the **transition probability matrix** $P = p_{ij}, i, j = 0, 1, \dots$. Note $1 = \sum_j P[X_{n+1} = j | X_n = i] = \sum_j p_{ij}$. n -step transition probability matrix: $P(n) = P^n$. State pmf at time n : $\mathbf{p}(n) = \mathbf{p}(0)P^n, n = 1, 2, \dots$. X_n **irreducible** if $\forall i, j \exists n \geq 0$ st $P[X_n = j | X_0 = i] > 0$. State i has **period** $k = \text{gcd}\{n \text{ st } p_{ii}(n) > 0\}$. Even if a state i has period k , it may not be possible to return to i in k steps. Irreducible X_n **aperiodic** if all states in its single class have period 1. **Stationary state pmf:** $\pi_j = \sum_i p_{ij} \pi_i$ st $\sum_i \pi_i = 1$. MC is stationary if $\mathbf{p}(0) = \boldsymbol{\pi}$, as $\mathbf{p}(n) = \boldsymbol{\pi}P^n = \boldsymbol{\pi} \forall n$.

Consider a **system** in which an input signal $x(t)$ is mapped to an output signal $y(t)$ by the transformation $y(t) = T[x(t)]$. System is **linear** if $T[\alpha x_1(t) + \beta x_2(t)] = \alpha T[x_1(t)] + \beta T[x_2(t)]$. It is **time-invariant** if the response to $x(t - \tau)$ is $y(t - \tau)$. **Impulse response** of an LTI system is $h(t) = T[\delta(t)]$ for $\delta(t)$ a unit delta function applied at $t = 0$. For arbitrary input $x(t)$, $y(t) = h(t) * x(t) = \int_{\mathbb{R}} h(s)x(t - s)ds = \int_{\mathbb{R}} h(t - s)x(s)ds$. **Transfer function** $H(f) = \mathcal{F}\{h(t)\} = \int_{\mathbb{R}} h(t)e^{-j2\pi ft}dt$.

For X_t, Y_t disc.-time, zero-mean, jointly WSS and $Y_t = \sum_{\beta=t-\alpha}^{t+\beta} h_{t-\beta} X_\beta = \sum_{\beta=-b}^a h_\beta X_{t-\beta}$, the **optimum filter** for estimating Z_t satisfies $R_{Z,X}(m) = \sum_{\beta=-b}^a h_\beta R_X(m - \beta)$ for $-b \leq m \leq a$ and has $E[(Z_t - Y_t)^2] = R_Z(0) - \sum_{\beta=-b}^a h_\beta R_{Z,X}(\beta)$. If we wish to predict Z_n in terms of $Z_{n-1}, Z_{n-2}, \dots, Z_{n-p}$, we have a **linear prediction problem**: $X_\alpha = Z_\alpha$ and $R_Z(m) = \sum_{\beta=1}^p h_\beta R_Z(m - \beta)$ for $m \in \{1, \dots, p\}$, or $[R_Z(1), R_Z(2), \dots, R_Z(p)]^T = \mathbf{R}_Z[h_1, h_2, \dots, h_p]^T$, where

$$\mathbf{R}_Z = \begin{bmatrix} R_Z(0) & R_Z(1) & R_Z(2) & \dots & R_Z(p-1) \\ R_Z(1) & R_Z(0) & R_Z(1) & \dots & R_Z(p-2) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & R_Z(1) \\ R_Z(p-1) & \cdot & \cdot & R_Z(1) & R_Z(0) \end{bmatrix}$$

and $E[e_n^2] = R_Z(0) - \sum_{\beta=1}^p h_\beta R_Z(\beta)$.

$X(t)$ has **independent increments** if for any k and any choice of sampling instants $t_1 < t_2 < \dots < t_k$, the RVs $X(t_2) - X(t_1), X(t_3) - X(t_2), \dots, X(t_k) - X(t_{k-1})$ are independent RVs. $X(t)$ has indep. increments $\Rightarrow X(t)$ Markov. If increments of the same length have the same distribution regardless of time origin, i.e., if $P[X_{n'} - X_n = y] = P[X_{n'-n} = y]$, X_n has **stationary increments**.

$X(t)$ **Gaussian** if the samples $X_1 = X(t_1), X_2 = X(t_2), \dots, X_k = X(t_k)$ are jointly Gaussian RVs $\forall k, t_1, \dots, t_k$. Joint pdf: $f_{X_1, \dots, X_k}(x_1, \dots, x_k) = \exp\{-\frac{1}{2}(\mathbf{x} - \mathbf{m})^T K^{-1}(\mathbf{x} - \mathbf{m})\} (2\pi)^{-\frac{k}{2}} |K|^{-\frac{1}{2}}$, where $\mathbf{m} = [m_X(t_1), \dots, m_X(t_k)]^T$, $K_{ij} = C_X(t_i, t_j)$. The sum of Gaussian processes is Gaussian, with joint pdf $f_{S_{n_1}, S_{n_2}}(y_1, y_2) = f_{S_{n_2-n_1}}(y_2 - y_1) f_{S_{n_1}}(y_1)$.

X_n **iid** if it consists of a sequence of iid RVs with common cdf $F_X(x)$, mean m , variance σ^2 . Joint cdf: $F_{X_1, \dots, X_k}(x_1, \dots, x_k) = F_X(x_1)F_X(x_2)\dots F_X(x_k)$. Mean: $m_X(n) = m \forall n$. Autocovariance: $C_X(n_1, n_2) = \sigma^2 \delta_{n_1, n_2}$ for δ_{ij} the Kronecker delta. Autocorrelation: $R_X(n_1, n_2) = C_X(n_1, n_2) + m^2$. iid process is Markov.

Bernoulli process I_n is a sequence of independent Bernoulli RVs with mean p . $m_I(n) = p$, $\text{var}[I_n] = p(1-p)$. Bernoulli process is iid.

Sum process $S_n = X_1 + X_2 + \dots + X_n = S_{n-1} + X_n$ where $S_n = 0$. S_n is Markov and ISI: $P[S_{n_1} = y_1, S_{n_2} = y_2, \dots, S_{n_k} = y_k] = P[S_{n_1} = y_1]P[S_{n_2-n_1} = y_2 - y_1] \dots P[S_{n_k-n_{k-1}} = y_k - y_{k-1}]$. $m_S(n) = E[S_n] = nE[X] = nm$. $\text{var}[S_n] = n\text{var}[X] = n\sigma^2$. $C_S(n, k) = \min(n, k)\sigma^2$.

Random step. Let $D_n = 2I_n - 1$, where I_n is the Bernoulli process. Then $D_n = 1$ if $I_n = 1$ and $D_n = -1$ if $I_n = 0$. $m_D(n) = 2p - 1$ and $\text{var}[D_n] = 4p(1-p)$. D_n is iid.

Binomial counting process. Let the I_i be a Bernoulli process with mean p , and let S_n be the corresponding sum process. S_n is the counting process that gives the number of successes in the first n Bernoulli trials, and is a binomial RV with parameters n and p , $P[S_n = j] = {}_n C_j p^j (1-p)^{n-j}$, for $0 \leq j \leq n$, and 0 otherwise. $m_S(n) = np$, $\text{var}[S_n] = np(1-p)$. S_n is ISI. Joint pmf at times n_1, n_2 : $P[S_{n_1} = y_1, S_{n_2} = y_2] = P[S_{n_1} = y_1]P[S_{n_2-n_1} = y_2 - y_1] = {}_{n_2-n_1} C_{y_2-y_1} p^{y_2-y_1} (1-p)^{n_2-y_2}$.

Random walk. Let D_n be the random step process and S_n be the corresponding sum process. S_n is a one-dimensional random walk process, with pmf $P[S_n = 2k - n] = {}_n C_k p^k (1-p)^{n-k}$ for $k \in \{0, 1, \dots, n\}$. $C_S(n, k) = \min(n, k)4p(1-p)$. Random walk is ISI.

Poisson process. Consider a situation in which events occur at random instants of time at an average rate of λ events per second, and $N(t)$ is the number of event occurrences in $[0, t]$. $P[N(t) = k] = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$, $k = 0, 1, \dots$; $N(t)$ is the Poisson

process and is ISI: $P[N(t_1) = i, N(t_2) = j] = P[N(t_1) = i]P[N(t_2) - N(t_1) = j - i] = P[N(t_1) = i]P[N(t_2 - t_1) = j - i]$. $m_N(t) = \text{var}[N(t)] = \lambda t$; $C_N(t_1, t_2) = \lambda \min(t_1, t_2)$. Interevent time T is an exponential RV with parameter λ : $P[T > t] = e^{-\lambda t}$; $E[T] = \frac{1}{\lambda}$, $\text{var}[T] = \frac{1}{\lambda^2}$. Interevent times form an iid sequence of exponential RVs. For T_j the iid exponential interarrival times, if $S_n = \sum_{j=1}^n T_j$ (the time at which the n th event occurs in the Poisson process), then $f_{S_n}(y) = \frac{(\lambda y)^{n-1}}{(n-1)!} \lambda e^{-\lambda y}$ for $y \geq 0$ (Erlang RV). Poisson process is Markov. Arrivals "at random": given only one arrival occurred in $[0, t]$, at time X , let $N(x)$ be the number of events up to time x ($0 < x < t$) and let $N(t) - N(x)$ be the increment in the interval $(x, t]$. Then $P[X \leq x] = P[N(x) = 1 | N(t) = 1] = \frac{x}{t}$; the arrival time of the given arrival is uniformly distributed in $[0, t]$.

Random telegraph signal. $X(t)$ assumes values $\{-1, 1\}$, and $X(0)$ is either with probability $\frac{1}{2}$. $X(t)$ changes polarity with each occurrence of an event in a Poisson process with rate α . $E[X(t)] = 0$, $\text{var}[X(t)] = 1$, $C_X(t_1, t_2) = \exp\{-2\alpha|t_2 - t_1|\}$. Is Markov.

Wiener process. $X(t)$ is a cont.-time symmetric random walk. $E[X(t)] = 0$, $\text{var}[X(t)] = \alpha t$, $C_X(t_1, t_2) = \alpha \min(t_1, t_2)$. Is Gaussian, Markov. Is the mean-square integral of white Gaussian noise process.

White Gaussian noise. $R_X(t_1, t_2) = \alpha \delta(t_1 - t_2)$. Is the mean square derivative of the Wiener process.

Fourier transform. For $G(f) = \mathcal{F}\{g(t)\} = \int_{\mathbb{R}} g(t)e^{-j2\pi ft} dt$, $g(t) = \mathcal{F}^{-1}\{G(f)\} = \int_{\mathbb{R}} G(f)e^{j2\pi ft} dt$. Linearity: $\mathcal{F}\{ag_1(t) + bg_2(t)\} = aG_1(f) + bG_2(f)$. Time scaling: $\mathcal{F}\{g(at)\} = G(f/a)/|a|$. Duality: If $\mathcal{F}\{g(t)\} = G(f)$, then $\mathcal{F}\{G(t)\} = g(-f)$. Time shifting: $\mathcal{F}\{g(t-t_0)\} = G(f)e^{-j2\pi ft_0}$. Frequency shifting: $\mathcal{F}\{g(t)e^{j2\pi f_0 t}\} = G(f - f_0)$. Differentiation: $\mathcal{F}\{g'(t)\} = j2\pi f G(f)$. Integration: $\mathcal{F}\{\int_{-\infty}^t g(s) ds\} = \frac{G(f)}{j2\pi f} + \frac{1}{2}G(0)\delta(f)$. Multiplication in time: $\mathcal{F}\{g_1(t)g_2(t)\} = G_1(f) * G_2(f)$. Convolution in time: $\mathcal{F}\{g_1(t) * g_2(t)\} = G_1(f)G_2(f)$.

Fourier transform pairs.

$g(t)$	$G(f)$
$1, t \in [-T, T]; 0$ otherwise	$\frac{2T}{2\pi fT} \sin 2\pi fT$
$\frac{2W}{2\pi Wt} \sin 2\pi Wt$	$1, t \in [-W, W]; 0$ otherwise
$1 + \frac{1}{T}, t \in [-T, 1]; 1 - \frac{1}{T}, t \in [1, T]$	$T \left(\frac{\sin \pi fT}{\pi fT} \right)^2$
$\delta(t)$	1
1	$\delta(t)$